# Minimum Angular Vibration Design of Airborne Electro-Optical Packages

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**Low-level angular vibration disturbances to airborne electro-optical packages can sometimes severely degrade the accuracy of such systems. This paper is an investigation into the possibility of minimizing the angular vibration response of selected points on an optical bench by redistributing the mass. The governing equations are presented in a form suitable for application of the Pontryagin maximum principle with the beam cross-sectional area playing the role of the control function. Angular vibration reduction at the ends of a pinned beam of an order of magnitude are demonstrated.**

## Nomenclature

$A(x)$ = variable cross-sectional area of the simply supported beam
$A_i$ = arbitrary mode shape constants ($i = 1,2,3,\ldots,12$)
$C$ = arbitrary constant
$E$ = Young's modulus
$F$ = viscoelastic damping constant
$\bar{f}(\bar{w},u,x)$ = right-hand side of state variable equations
$H(\bar{w},x,u,\bar{\lambda})$ = Hamiltonian
$h_i(\eta)$ = impulse response function for the $i$th mode
$k$ = constant in the optimal area distribution function
$L$ = length of the simply supported beam
$M_0$ = maximum total mass of the optical bench
$q_i(t)$ = generalized coordinate for the $i$th mode
$Q_{nk}$ = parameter showing cross-coupling between modes
$R(x)$ = spatial distribution of the excitation to the optical bench
$R_n$ = amplitude of the generalized force
$\langle R_n R_k \rangle$ = spatial correlation of the excitation
$S(x,\omega)$ = angular vibration power spectral density (PSD) function
$u$ = control function $= A(X)$
$v(r)$ = transformed variable in Eqs. (27) and (31)
$\bar{W}$ = vector of state variables in the Pontryagin maximum (minimum) principle
$x$ = position along the beam: independent variable in the Pontryagin maximum (minimum) principle
$y(x,t)$ = beam transverse forced response measured relative to the support motion
$\partial y/\partial x$ = beam angular forced response
$\langle(\partial y/\partial x)^2\rangle$ = mean-square angular vibration response
$\ddot{Z}_1(t)$ = left support random acceleration
$\ddot{Z}_2(t)$ = right support random acceleration
$\ddot{Z}_0$ = constant power spectral density of $Z(t)$
$\beta$ = eigenfunction parameter $= \sqrt[4]{\rho\omega^2/E\kappa^2}$
$\delta$ = small parameter used in the perturbation technique
$\epsilon$ = minimum allowable beam cross-sectional area
$\eta$ = dummy variable of integration
$\theta_u$ = rms angular vibration of the uniform beam
$\theta_0$ = rms angular vibration of the optimum beam
$\kappa$ = beam cross-section radius of gyration
$\bar{\lambda}$ = vector of Lagrange multipliers
$\xi$ = damping ratio for the $i$th mode $= \omega_i F/2E$
$\rho$ = mass density
$\phi_i(x)$ = $i$th mode shape
$\omega_i$ = $i$th natural frequency

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## I. Introduction

AIRBORNE electro-optical packages are very sensitive to the low-level angular vibration disturbance transmitted to the optical path by the airframe. In fact, the vibration level of the airframe could be quite low considering traditional aircrew tolerances and structural strength requirements but still be large enough to seriously degrade the accuracy of the electro-optical system by exciting the angular vibration of the optical bench. The angular vibration disturbance is defined as the slope of the optical bench deflection with respect to the spatial coordinate, or the rotary deflection. This paper is concerned with the design of an optical bench in such a way as to minimize the angular vibratory response of the ends of the bench.

Optimal structural design is a well-developed discipline, and it is beyond the scope of this paper to provide a rigorous survey of the numerous approaches available. Reference 1 is such a survey, and makes a broad distinction between two general approaches to optimal structural design. One approach is to formulate a discrete structural model and utilize one of a number of algorithms available for direct numerical searches for an optimum design on the digital computer. Reference 2 follows this general approach for seeking the minimum weight under constraints on the natural frequencies. Such approaches frequently involve a significant computational effort in solving for the structural natural frequencies. The other general approach which is more applicable to the angular vibration minimization problem is to transform the governing structural equations into an optimal control problem with a structural property playing the role of a control function. Reference 3 utilizes that approach to solve for the thickness distribution of a turbine disk which will maximize a linear combination of the natural frequencies. Although this approach has only been used for relatively simple systems, it can be used in the angular vibration minimization problem. The interested reader should consult Ref. 1 for a survey of various possible approaches in optimal structural design.

The minimum angular vibration design of electro-optical packages has some considerations which are different from other structural optimization problems. Since the optical package would probably be installed inside the airframe, aerodynamic instability is not of concern. In this application, the vibratory response of the airframe itself is the excitation source of the optical bench. It is the angular motion of the system which is of primary concern, since this disturbance is the most harmful to the accuracy of an optical train, and minimum *angular* vibration design has not yet been considered. The vibration disturbance is a random response since the airframe is responding to aerodynamic turbulence or jet engine noise. It is of secondary importance to minimize the weight of the electro-optical system, as long as it is compatible with the airframe. Therefore the total weight is a constraint in the formulation.

The approach to be used in attempting to design an optical bench for a minimum angular vibration response will be to consider a base-excited simply supported beam with the cross-sectional area variable. The mean-square angular response of the beam ends is calculated from the natural frequencies and normal mode functions. The ends of the beam are used since that would likely be the locations of steering mirrors on the optical bench. The form of the resulting equations is suitable for application of the Pontryagin maximum principle with the mean-square angular vibratory response the quantity to be minimized. The variable cross-sectional area of the beam plays the role of a control function. The mass of the airframe is assumed to be much greater than the mass of the electro-optical system so that redistributing the mass will not affect airframe vibration. It may be observed that minimizing the mean-square vibratory response is equivalent to maximizing a linear combination of the natural frequencies as in Ref. 3.

## II. Analysis

The optical bench is modeled as a pinned beam, where the airframe excites the system through the installation points as demonstrated in Fig. 1. It is assumed that the radius of gyration of the beam cross section is a constant, but the area is allowed to vary along the length of the beam. Since the forced response is the quantity under study, it is necessary to include damping in the analysis. Common damping models used in structural analysis are the viscous model and the complex modulus model. However, the viscous model does not permit separation of variables in the equation of motion given below, and the complex modulus model predicts that all the resonant peaks are of the same height. Since neither of these observations is consistent with practical observations, a viscoelastic damping model is assumed for the optical bench. That model is consistent with this application and allows the damping to be redistributed as well as the mass. In addition, simple laboratory experiments can be conducted to show that for most materials the viscoelastic damping model is a good approximation to actual random vibration measurements for the first few natural frequencies. Finally, the viscous model has been shown to yield a diverging stress in the random excitation of a beam. The viscoelastic model does not have this undesirable property. According to the viscoelastic damping model, the stress in a material is proportional to strain and strain rate. Therefore the equation of motion for the base-excited variable area beam of Fig. 1 is

$$E\kappa^2 \frac{\partial^2}{\partial x^2}\left[A(x)\frac{\partial^2 y}{\partial x^2} + A(x)\frac{F}{E}\frac{\partial^3 y}{\partial x^2 \partial t}\right] + \rho A(x)\frac{\partial^2 y}{\partial t^2} = \rho A(x) R(x) \ddot{Z} \tag{1}$$

The solution to Eq. (1) is assumed to be of the form

$$y(x,t) = \sum_{i=1}^{\infty} \phi_i(x) q_i(t) \tag{2}$$

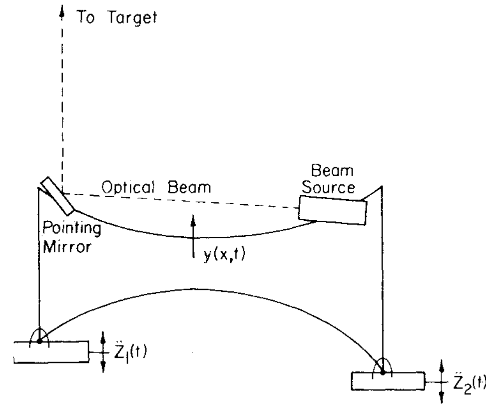


**Fig. 1 Pinned beam model of an airborne optical bench.**

The natural frequencies and normal mode functions are required in order to calculate $y(x,t)$ by Eq. (2). That may be done by setting the right-hand side of Eq. (1) to zero and separating variables. (See Sec. IV.) That result is

$$\frac{E\kappa^2}{\rho A(x)\phi_i(x)}\frac{\partial^2}{\partial x^2}\left[A(x)\frac{d^2\phi_i}{dx^2}\right] = \frac{-\ddot{q}_i}{q_i + 2\xi_i/\omega_i \dot{q}_i} = \omega_i^2 \tag{3}$$

The two parts to Eq. (3) are

$$\ddot{q}_i + 2\xi_i\omega_i\dot{q}_i + \omega_i^2 q = 0 \tag{4}$$

$$E\kappa^2 \frac{\partial^2}{\partial x^2}\left[A(x)\frac{d^2\phi_i}{dx^2}\right] - \rho A(x)\omega_i^2\phi_i(x) = 0 \tag{5}$$

Equation (4) just gives the well-known decaying sinusoid form of the vibratory response, while Eq. (5) is the governing equation of the optimal design to be conducted in choosing $A(x)$. In fact, Eq. (5) is analogous to Eq. (8) of Ref. 3. However, before the optimal choice of $A(x)$ is made, it is necessary to derive the form for the random angular vibration response.

In Eq. (1), $R(x)$ is the spatial distribution of the base excitation and has been derived in the Appendix. (See Ref. 4 for a similar forced response analysis.) Then the general solution is

$$y(x,t) = \sum_{i=1}^{\infty} \phi_i(x) R_i \int_0^t \ddot{Z}(t-\eta) h_i(\eta) d\eta \tag{6}$$

In Eq. (6), $R_i$ is the generalized force and is derived in the Appendix. Equation (6) may be differentiated with respect to $x$ to solve for the beam angular vibration, and assuming the airframe acceleration is white noise, the angular vibration power spectral density (PSD) may be computed (see Refs. 5 and 6 for a derivation of the random response of a viscously damped beam):

$$S(x,\omega) = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \frac{d\phi_k}{dx}\frac{d\phi_n}{dx} \times \frac{\langle R_n R_k \rangle \ddot{Z}_0}{[\omega_n^2 - \omega^2 + j2\xi_n\omega_n\omega][\omega_k^2 - \omega^2 - j2\xi_k\omega_k\omega]} \tag{7}$$

In Eq. (7), $\langle R_n R_k \rangle$ is the spatial correlation of the excitation, in this case a deterministic function which can be computed. Equation (7) differs from the vibration PSD functions of Refs. 5 and 6 in that the damping model is viscoelastic. Although mean-square stress for the viscous damping model is known to diverge,[7] the mean-square angular response of the viscously damped beam was shown to converge.[6] The viscoelastic damping model improves that rate of convergence. Because of this, Eq. (7) is believed to be a consistent representation of the angular vibration PSD including all modes.

The mean square of a random process is defined as the integral of the PSD-frequency curve. That was calculated in Ref. 6 for the viscous damping model, and it was concluded that for the uniform area simply supported beam, the cross-coupling between modes has a negligible effect on the mean-square value. Since the viscoelastic damping model is used here, the particular form for the mean-square angular vibration is different from Ref. 6. Using contour integration, the mean-square angular vibration response is

$$\left\langle \left(\frac{\partial y}{\partial x}\right)^2 \right\rangle = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left\{ \frac{\mathrm{d}\phi_k}{\mathrm{d}x} \frac{\mathrm{d}\phi_n}{\mathrm{d}x} \langle R_n R_k \rangle \ddot{Z}_0 (\xi_n \omega_n + \xi_k \omega_k) \pi \right.$$
$$\div 2\omega_n^2 (1-\xi_n^2)(\xi_n\omega_n + \xi_k\omega_k)^2$$
$$\left. + \tfrac{1}{2} [\omega_n^2 - \omega_k^2 - 2\xi_n\omega_n(\xi_n\omega_n + \xi_k\omega_k)]^2 \right\} \tag{8}$$

When $\xi_k\omega_k = \xi_n\omega_n$ in Eq. (8), the damping model would be viscous and Eq. (8) agrees with Eq. (11) of Ref. 6. Considerable insight can be gained into the contributions of the higher-order modes by evaluating Eq. (8) for the first few modes, showing that the first mode dominates the mean square. This is done in Sec. V. Furthermore, the angular mean-square vibration response is decreased when the natural frequencies are increased. Thus the problem of minimizing the angular vibration is equivalent to maximizing the natural frequencies as in Ref. 3. The Pontryagin maximum principle may be used to minimize the angular mean-square response while limiting the mass of the optical bench. That optimization problem will be considered in Sec. III.

## III. Optimal Area Distribution

deSilva[3] solved for an optimal thickness distribution for turbine blades to maximize a linear combination of natural frequencies. That approach was to cast the equations of motion in the form of an optimal control problem where the turbine blade thickness played the role of a control variable in the Pontryagin maximum (minimum) principle, and the spatial coordinate is the independent variable. The Pontryagin maximum (minimum) principle is briefly stated here for convenience.

For the angular vibration minimization problem, the index of performance is defined as

$$J = \left\langle \left(\frac{\partial y}{\partial x}\right)^2 \right\rangle \tag{9}$$

Then, following the approach of Ref. 8, Eq. (5) is written in the form

$$\frac{\mathrm{d}\bar{w}}{\mathrm{d}x} = \bar{f}(\bar{w}, u, x) \tag{10}$$

Then a Hamiltonian is defined as

$$H(\bar{w}, x, u, \bar{\lambda}) = \bar{\lambda}^T \bar{f}(\bar{w}, u, x) \tag{11}$$

where the constraint on the total mass of the beam is

$$\int_0^\ell \rho A(x)\,\mathrm{d}x \le M_0 \tag{12}$$

and the necessary conditions for a minimum are[8]:

$$\frac{\partial H}{\partial w_i} = \frac{\mathrm{d}\lambda_i}{\mathrm{d}x} \tag{13}$$

$$\frac{\partial H}{\partial u} = 0 \tag{14}$$

deSilva[3] applied Eqs. (13) and (14) to the optimal vibrational design of a turbine disk, but Carmichael[9] showed that the resulting optimal control problem was singular. This means that the solution may not necessarily be a minimum or maximum, and Carmichael[9] gave an additional test to check singularity. The formulation given here is modified from that given by deSilva[3] according to Carmichael's[9] work to avoid the singular optimal control problem. Equation (5) is transformed into state variables below:

$$w_1 = \phi(x), \quad \frac{\mathrm{d}w_1}{\mathrm{d}x} = w_2, \quad \frac{\mathrm{d}w_2}{\mathrm{d}x} = w_3, \quad \frac{\mathrm{d}w_3}{\mathrm{d}x} = w_4$$

$$\frac{\mathrm{d}w_4}{\mathrm{d}x} = \frac{\rho\omega^2}{E\kappa^2} w_1 - \frac{z}{u}\frac{\mathrm{d}u}{\mathrm{d}x} w_4 - \frac{1}{u}\frac{\mathrm{d}^2u}{\mathrm{d}x^2} w_3, \quad u = A(x) \tag{15}$$

Then the Hamiltonian [Eq. (11)] is

$$H(\bar{\lambda}, \bar{w}, x, u) = \lambda_1 w_2 + \lambda_2 w_3 + \lambda_3 w_4 + \lambda_4 \left[ \frac{\rho\omega^2}{E\kappa^2} w_1 - \frac{2}{u}\frac{\mathrm{d}u}{\mathrm{d}x} w_4 - \frac{1}{u}\frac{\mathrm{d}^2u}{\mathrm{d}x^2} w_3 \right] \tag{16}$$

Then the necessary conditions for a minimum yield

$$\frac{\mathrm{d}\lambda_1}{\mathrm{d}x} = -\lambda_4 \frac{\rho\omega^2}{E\kappa^2}, \quad \frac{\mathrm{d}\lambda_2}{\mathrm{d}x} = -\lambda_1$$

$$\frac{\mathrm{d}\lambda_3}{\mathrm{d}x} = -\lambda_2 + \frac{\lambda_4}{u}\frac{\mathrm{d}^2u}{\mathrm{d}x^2}, \quad \frac{\mathrm{d}\lambda_4}{\mathrm{d}x} = -\lambda_3 + \frac{2\lambda_4}{u}\frac{\mathrm{d}u}{\mathrm{d}x} \tag{17}$$

From Eq. (14),

$$\frac{\partial H}{\partial u} = \lambda_4 \left[ \frac{2w_4}{u^2}\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{w_3}{u^2}\frac{\mathrm{d}^2u}{\mathrm{d}x^2} \right] = 0 \tag{18}$$

This means that either $\lambda_4$ is zero or

$$\frac{2w_4}{u^2}\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{w_3}{u^2}\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = 0 \tag{19}$$

If $\lambda_4$ were zero, the solution would not involve $A(x)$ and this would be the trivial solution. Therefore Eq. (19) must hold, and Eqs. (17) represent a set of nonlinear differential equations whose solution can be approximated by the perturbation technique after Ref. 3. This assumes that

$$\lambda_i = \sum_{j=0}^{\infty} \lambda_{ij}\delta^j$$

where $\delta$ is a small parameter.

If $\lambda_4$ is small and constant, then $\mathrm{d}\lambda_4/\mathrm{d}x = 0$, and

$$0 = -\lambda_3 + 2\lambda_4 \frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}x}$$

By Eqs. (17), $\lambda_3$ cannot be zero and the differential equation for the optimum area distribution is

$$\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}x} = k \tag{20}$$

where $|k|$ is very large.

From Eq. (19),

$$k = -\frac{1}{2A^*}\frac{\mathrm{d}^2A^*}{\mathrm{d}x^2}\frac{w_3^*}{w_4^*} \tag{21}$$

In Ref. 3 it is proven that the constant $k$ is very large for the optimum turbine disk. It can easily be shown that this

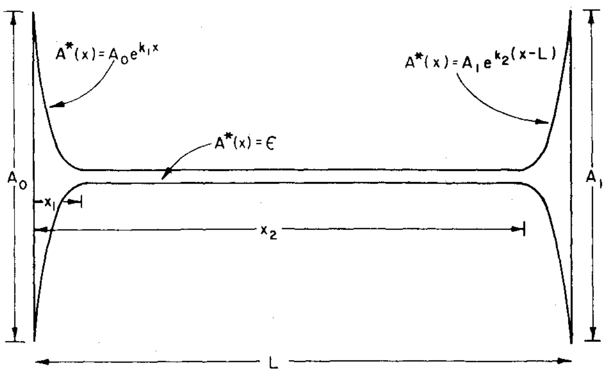


**Fig. 2 Optimum distribution of the pinned beam to minimize the angular vibration of the ends.**

problem is the same as the optimum disk problem, and although Eq. (21) appears to be a function of $x$, it is really a constant over the entire beam for some particular optimum $A$, designated by *. That development will not be repeated here. The solution to Eq. (20) is

$$A(x) = Ce^{kx} \tag{22}$$

With $k$ large, $A(x)$ could become negligible, so an additional physical constraint is necessary:

$$A(x) \geq \epsilon$$

where $\epsilon$ is the minimum allowable cross-sectional area. With that additional constraint, the optimal area distribution is (see Fig. 2):

$$\begin{aligned} A^*(x) &= A_0 \exp[k_1(x-x_1)] \quad 0 \leq x \leq x_1 \quad (k_1 \text{ negative}) \\ &= \epsilon \qquad\qquad x_1 \leq x \leq x_2 \\ &= A_1 \exp[-k_2(x-L)] \quad x_2 \leq x \leq L \quad (k_2 \text{ positive}) \end{aligned} \tag{23}$$

Since $A^*(x)$ reaches its minimum at $x=x_1$ and $x_2$, the values for $k_1$ and $k_2$ are

$$k_1 = \frac{1}{x_1} \log \frac{\epsilon}{A_0} \tag{24}$$

$$k_2 = \frac{1}{L-x_2} \log \frac{\epsilon}{A_1} \tag{25}$$

The qualities $x_1$ and $x_2$ are to be chosen on the basis of design constraints and the maximum mass constraint.

The choice for $x_1$ and $x_2$ is based on the feasible region demonstrated in Fig. 3. The feasible region is shown as the shaded region. The level curves are found from Eqs. (12) and (23):

$$\begin{aligned} M_0 &= \rho(A_0/k_1)(\exp[k_1x_1]-1) \\ &+ \rho\epsilon(x_2-x_1)\rho(A_1/k_2)\exp[k_2(L-x_2)]-1 \end{aligned} \tag{26}$$

Suitable values for $x_1$ and $x_2$ may be chosen using Fig. 3 and engineering considerations of the maximum allowable structural mass $M_0$. Figure 3 should be viewed as a design tool; for every design choice of $x_1$ and $x_2$, given $A_0$, $A_1$, and $\epsilon$, a different structural mass results, and that mass must be checked against the maximum.

The validity of the assumption that $\lambda_4$ is negligibly small depends on the assumption that $\epsilon$ is small, which would force $k_1$, $k_2$ to be large. In the limit as $\epsilon \to 0$, $k_1$, $k_2 \to \infty$. Now Eq. (23) can be substituted back into Eq. (5) to solve for the

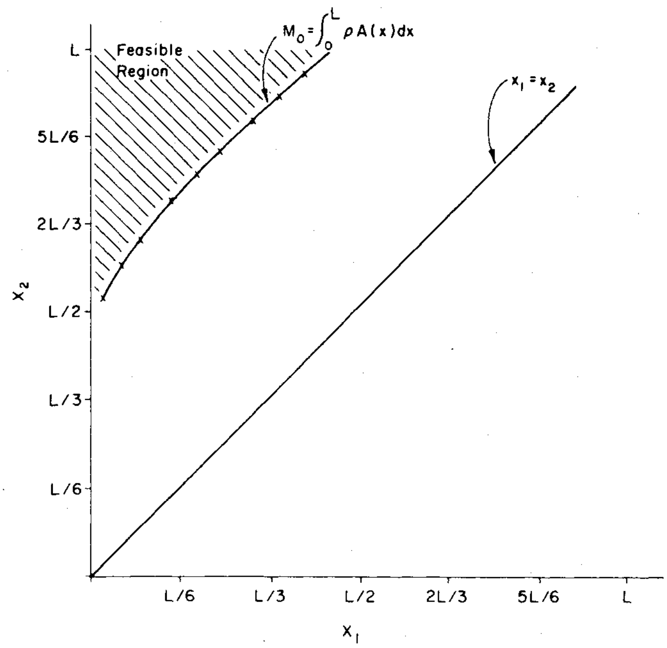


**Fig. 3 Feasible region of the minimum angular vibration design problem under constrained mass.**

optimal natural frequencies and corresponding mode shapes. That will be done in Sec. IV.

## IV. Optimal Mode Shapes and Natural Frequencies

The mode shape functions and natural frequencies will be determined piecewise by substituting Eq. (23) back into Eq. (5). This approach is after deSilva.[3] (See Fig. 2.)

**Region 1:** $0 \leq x \leq x_1$

Transform the variables by substituting $r = -k_1(x-x_1)$, $k_1$ large and negative, into Eq. (5) and the result is

$$\frac{d^4\phi}{dr^4} - 2\frac{d^3\phi}{dr^3} + \frac{d^2\phi}{dr^2} - \frac{\rho\omega^2}{E\kappa^2k_1^4} = 0 \tag{27}$$

Now assume $\phi(r) = e^{r/2}v(r)$ and Eq. (27) becomes

$$\frac{d^4v}{dr^4} - \frac{1}{2}\frac{d^2v}{dr^2} + \left[\frac{1}{16} - \frac{\rho\omega^2}{E\kappa^2k_1^4}\right]v = 0 \tag{28}$$

Solutions to Eq. (28) are

$$v(r) = A_1e^{s_1r} + A_2e^{-S_1r} + A_3e^{S_2r} + A_4e^{-S_2r}$$

$$S_1 = \sqrt{1/4 + \sqrt{\rho\omega^2/E\kappa^2k_1^4}}$$

$$S_2 = \sqrt{1/4 - \sqrt{\rho\omega^2/E\kappa^2k_1^4}}$$

Back-substituting for $u(r)$ and $r$ gives

$$\begin{aligned} \phi_1(x) &= A_1\exp[-(s_1+1/2)k_1(x-x_1)] \\ &+ A_2\exp[-(-s_1+1/2)k_1(x-x_1)] \\ &+ A_3\exp[-(s_2+1/2)k_1(x-x_1)] \\ &+ A_4\exp[-(-s_2+1/2)k_1(x-x_1)] \end{aligned} \tag{29}$$

**Region 2:** $x_1 \leq x \leq x_2$

$$E\kappa^2\epsilon\frac{d^2w}{dx^4} - \omega^2\rho\epsilon w = 0$$

If $\beta^4 = \rho\omega^2 / E\kappa^2$,

$$\phi_2(x) = A_5\cos(\beta x) + A_6\sin(\beta x) + A_7\cosh(\beta x) + A_8\sinh(\beta x) \quad (30)$$

**Region 3:** $x_2 \le x \le L$

Transform the variables by substituting $r = -k_2(x-L)$, where $k_2$ is large and positive, into Eq. (5) and the result is

$$\frac{d^4\phi}{dr^4} - 2\frac{d^3\phi}{dr^3} + \frac{d^2\phi}{dr^2} - \rho\frac{\omega^2}{E\kappa^2 k_2^4}\phi = 0 \quad (31)$$

Now, assuming $\phi(r) = e^{+r/2}v(r)$,

$$\frac{d^4 v}{dr^4} - \frac{1}{2}\frac{d^2 v}{dr^2} + \left[\frac{1}{16} - \rho\frac{\omega^2}{E\kappa^2 k_2^4}\right]v = 0 \quad (32)$$

Solutions to Eq. (31) are of the form

$$v(r) = A_9 e^{s_3 r} + A_{10}e^{-s_3 r} + A_{11}e^{s_4 r} + A_{12}e^{-s_4 r}$$

where

$$s_3 = \sqrt{1/4 + \sqrt{\rho\omega^2 / E\kappa^2 k_2^4}}$$

$$s_4 = \sqrt{1/4 - \sqrt{\rho\omega^2 / E\kappa^2 k_2^4}}$$

Back-substituting for $v(r)$ and $r$ gives

$$\begin{aligned}\phi_3(x) &= A_9\exp[(s_3 + 1/2)k_2(x-L)] \\ &+ A_{10}\exp[(-s_3 + 1/2)k_2(x-L)] \\ &+ A_{11}\exp[(s_4 + 1/2)k_2(x-L)] \\ &+ A_{12}\exp[(-s_4 + 1/2)k_2(x-L)] \end{aligned} \quad (33)$$

There are 12 arbitrary constants to determine, and 12 boundary conditions: two boundary conditions at each of the ends are required, given by the geometry, and at $x_1$ and $x_2$ the displacement, slope, moment, and shear must be continuous. The resulting eigenvalue problem can then be solved by searching for the frequencies where the determinant of the coefficient matrix vanishes. An example problem will be investigated in Sec. V to determine the degree of angular vibration minimization possible.

## V. Discussion

The optimum area distribution for the simply supported beam to minimize angular vibration will now be examined for a particular beam. Nominal values for the properties of the beam are

| | |
|---|---|
| $E = 69.0 \times 10^9$ N/m | $k = 2.88 \times 10^{-3}$ m |
| $F = 2.79 \times 10^6$ N/m | $L = 0.6$ m |
| $\rho = 2.63 \times 10^3$ kg/m$^3$ | $x_1 = 0.05$ m |
| $A_0 = 0.25$ m$^2$ | $x_2 = 0.55$ m |
| $A_1 = 0.25$ m$^2$ | $\epsilon = 2.5 \times 10^{-5}$ m$^2$ |

The first few natural frequencies and angular mode shapes at the ends are summarized in Table 1. The mean-square angular vibration under white noise base acceleration is given by Eq. (8), and it can be seen from Table 2 that taking only the first mode gives a good approximation to the angular mean-square value.

**Table 1 Natural frequencies and modal deflections for symmetrical optimum and uniform beams**

| | Uniform | | Optimum | |
|---|---|---|---|---|
| $n$ | $\omega_n$ | $d\phi_n/dx$ | $\omega_n$ | $d\phi_n/dx$ |
| 1 | 404.42 | 5.236 | 495.12 | 0.367 |
| 2 | 1617.67 | 10.472 | 1991.62 | 1.289 |
| 3 | 3639.8 | 15.708 | 4208.87 | 1.947 |
| 4 | 6470.7 | 20.944 | 7205.43 | 1.501 |

**Table 2 Effect of modal coupling on $Q_{nk}$ for optimum and uniform beams**

| $n, k$ | Uniform | Optimum |
|---|---|---|
| 1, 1 | $1.45 \times 10^{-6}$ | $6.47 \times 10^{-7}$ |
| 1, 2 | $5.86 \times 10^{-11}$ | $3.86 \times 10^{-11}$ |
| 1, 3 | $9.94 \times 10^{-12}$ | $7.46 \times 10^{-12}$ |
| 2, 2 | $5.67 \times 10^{-9}$ | $2.47 \times 10^{-9}$ |
| 2, 3 | $1.75 \times 10^{-11}$ | $1.42 \times 10^{-11}$ |
| 3, 3 | $2.20 \times 10^{-10}$ | $1.23 \times 10^{-10}$ |

**Table 3 Spatial correlation of the excitation $\langle R_n R_k \rangle$**

| | Uniform | | Optimum | |
|---|---|---|---|---|
| $n, k$ | $\alpha = -1$ | $\alpha = 0$ | $\alpha = -1$ | $\alpha = 0$ |
| 1, 1 | 1.292 | 1 | 26.39613 | 111.414 |
| 2, 2 | 0.6696 | 0 | 1.9616 | $2.1255 \times 10^{-2}$ |
| 3, 3 | 0.50724 | 1 | 0.1579244 | 0.3833443 |
| 1, 2 | 0.948278 | 0 | 7.19574 | 1.539 |
| 1, 3 | 0.8095 | 1 | 2.041713 | 6.5353 |
| 2, 3 | 0.58279 | 0 | 0.55658 | $9.0266 \times 10^{-2}$ |

In Table 2, $\xi_1 = 0.01$ and $Q_{nk}$ is defined as

$$Q_{nk} = \pi(\xi_n\omega_n + \xi_k\omega_k) / \{2\omega_n^2(1-\xi_n^2)(\xi_n\omega_n + \xi_k\omega_k)^2 + 1/2[\omega_n^2 - \omega_k^2 - 2\xi_n\omega_n(\xi_n\omega_n + \xi_k\omega_k)]^2\}$$

This is consistent with the results given in Ref. 6, where the first mode dominated the angular vibration of the uniform simply supported beam. Neglecting all but the first mode, the angular mean-square response is approximately

$$\left\langle \left(\frac{\partial y}{\partial x}\right)^2 \right\rangle = \left(\frac{d\phi_1}{dx}\right)^2 \ddot{Z}_0 Q_{11}\langle R_1^2 \rangle$$

Using the spatial correlation function derived in the Appendix, Table 3 has been prepared to aid in calculating the approximate root-mean-square angular response. Table 3 contains two cases for the base excitation: when $\alpha = 0$ the base moves as a rigid body, and when $\alpha = -1$ one end of the beam is fixed. From Table 3 it is apparent that the spatial excitation depends on the vibration patterns of the primary structure to which the optical bench is to be attached.

The degree of angular vibration reduction achieved by the optimum beam therefore depends on how the optical bench is excited. Considering the case that the base moves as a rigid body, $\alpha = 0$,

$$\theta_u = 6.3 \times 10^{-3}\sqrt{\ddot{Z}_0} \qquad \theta_o = 3.1 \times 10^{-3}\sqrt{\ddot{Z}_0}$$

This case offers only about 50% reduction in the angular vibration response. For the case that one end is pinned,

$$\theta_u = 7.167 \times 10^{-3}\sqrt{\ddot{Z}_0} \qquad \theta_o = 1.517 \times 10^{-3}\sqrt{\ddot{Z}_0}$$

This case offers about 80% reduction in the angular vibration response.

It must be remembered that this solution for an optimum beam depends on the condition $A(x) \geq \epsilon$ where $\epsilon$ is a small number. It was noted in Sec. III that the assumption of $k$ being large depends on $\epsilon$ being small enough. Whether or not $\epsilon$ is small enough depends also on the boundary conditions of the beam, since for a clamped beam $\epsilon$ must be smaller than for the simply supported beam in order for the perturbation solution to be valid. For the optical bench, this just means that the midportion of the beam only serves to keep the ends aligned so that the electro-optical system can operate: the load is carried by the ends.

The significant reduction in angular vibration can be understood qualitatively as increasing the rotary inertia of the beam ends, and thus minimizing the angular response while at the same time increasing the natural frequencies. Also, the viscoelastic damping model chosen affects the reduction in angular vibration since it causes the rms angular response to depend on natural frequency to the fourth power. Some other damping model might not yield nearly as much angular vibration reduction. However, Ref. 6 showed that the first mode still dominates for the viscous damping model so these assumptions are not invalidated by the viscous model. Further analysis of this aspect could better be conducted after more is known concerning structural damping.

It should be pointed out that the optimum beam of Fig. 2 holds only with respect to angular vibration. No attempt was made to include considerations of strength or displacement limits, and such practical problems would have to be addressed in order to reduce this approach to a real design. In addition, such considerations might have a severe impact on the assumption expressed in Eq. (20) which could change the entire nature of the problem. Any future work in this area must include these aspects of this approach.

## VI. Conclusions

The optimum distribution of mass in a pinned, base-excited beam is determined by transforming the governing equations into an optimal control problem with the cross-sectional area playing the role of a control function. The solution to the resulting Pontryagin minimization problem was approximated by using the perturbation method. This approximate solution reveals that a significant reduction in angular mean-square response at the ends is possible by concentrating most of the beam mass near the ends. Necessary conditions for a minimum show that the solution is the local minimum.

The reduction in the angular vibration response is almost an order of magnitude. This reduction is accomplished both by increasing the structural stiffness and by increasing the rotary inertia of the ends. This amount of reduction is significant and this approach should be considered in minimizing angular disturbances to airborne electro-optical systems. Notice that by concentrating most of the mass of the optical bench at the ends, the role of the bench itself is being minimized, and the concept of a "benchless" optical package comes to mind.

## Appendix

The spatial distribution of the base excitation $R(x)$ in Eq. (1) may be determined by considering the relative motion of the two supports. The rigid body motion will therefore be

$$\ddot{y}_s = \ddot{Z}_1(1 - x/L) + \ddot{Z}_2 x/L$$

where $\ddot{y}_s$ is the rigid body motion, $\ddot{Z}_1$ the random vibration of the left support, and $\ddot{Z}_2$ the vibration of the right support.

The function $R(x)$ from Eq. (1) is

$$R(x) = (1 + \alpha x/L)$$

where $\ddot{Z}_2/\ddot{Z}_1 = 1 + \alpha$. Then the spatial correlation of the excitation used in Eq. (7) is

$$\langle R_n R_k \rangle = \frac{1}{L^2} \int_0^\alpha \phi_n(x)\left(1 - \frac{\alpha x}{L}\right)\mathrm{d}x \int_0^L \phi_k(x)\left(1 - \frac{\alpha x}{L}\right)\mathrm{d}x$$

For the uniform pinned beam,

$$R_n = \frac{1}{L}\int_0^\alpha \sin\frac{n\pi x}{L}\left(1 - \frac{x}{L}\right)\mathrm{d}x = \left[\frac{4 + n\pi}{2\pi n}\right]$$

For the optimum beam, Eqs. (29), (30), and (33) must be combined to give

$$R_n = \frac{1}{L}\left[\int_0^{x_1} \phi_1(x)\,\mathrm{d}x + \int_{x_1}^{x_2} \phi_2(x)\,\mathrm{d}x + \int_{x_2}^{L} \phi_3(x)\,\mathrm{d}x\right]$$

The above integral has been evaluated and the results are summarized in Table 3. The choice for $\alpha$ is somewhat arbitrary, so Table 3 gives results for $\alpha = -1$, corresponding to $\ddot{Z}_2 = 0$, and $\alpha = 0$, corresponding to $\ddot{Z}_2 = \ddot{Z}_1$.

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